Event-Based LQR with Integral Action

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***Abstract*— In this paper, a state-feedback linear-quadratic regulator (LQR) is proposed for event-based control of a linear system. An interesting property of LQRs is that an optimal response of the system can be obtained in accordance to some specifications, like the actuator limits. An integral action is also added in order to not only restrict the study to null stabilization but also to tracking. The idea is to consider an external control loop and stabilize the integral of the error between the measurement and a desired setpoint to track. However, an event-triggered integral can lead to important overshoots when the interval between two successive events becomes large. Therefore, an exponential forgetting factor of the sampling interval is proposed as a solution to avoid such problems. The whole proposal is tested on a real-time system (a gyroscope) in order to highlight its ability, the reduction of control updates and the respect to the actuator limits.**

INTRODUCTION

The consistently-used periodic fashion cannot be applied anymore in *embedded and networked systems* (with limited resources) and resource-aware implementations are hence required. In this context, recent works addressed alternative frameworks where the control law is event driven. Whereas the control law is computed and updated at the same rate regardless whether is really required or not in the classical time-triggered approach, the *event-based paradigm* relaxes the periodicity of computations and communications in calling for resources whenever they are indeed necessary (for instance when the dynamics of the controlled system varies). Typical event-detection mechanisms are functions on the variation of the state (or at least the output) of the system, like in [2], [5], [16], [14], [9], [11], [8]. Although event-based control is well-motivated, only few works report theoretical results about stability, convergence and performance. It has notably been shown in [3] that the control law can be updated less frequently than with a periodic scheme while still ensuring the same performance. Stabilization is analyzed in [19], [17], [12], [6], where the events are related to the variation of a Lyapunov function or the time derivative of a Lyapunov function (and consequently to the state too). In the latter case, the updates ensure the strict decrease of the Lyapunov function, and so is asymptotically stable the closed-loop system.

The setup suggested in the present paper is based on the seminal work in [12] originally developed for general non-linear systems, which yields an *event-based linear-quadratic regulator* (LQR) in the particular linear case, see [18], [4]. LQRs offer interesting properties, which most important is that an optimal response of the system can be obtained in accordance to the designer's specifications (such as the limitations of actuators for instance). They can also be methodologically applied whatever the order of the system, and are intrinsically stable. On the other hand, adding an *integral action* allows to track a given setpoint (instead of a classical null stabilization) and a better perturbation robustness [10]. The idea is to stabilize the controlled system as well as the integral of the measured error (the error between the measurement and a desired setpoint to track). However, the integral can lead to important overshoots in the event-based scheme because the interval between two successive events is no more bounded. This was discussed in [5] for the design of an event-based PI (proportional integral) controller, and several methods were suggested as solutions to avoid such problems. In particular, an *exponential forgetting factor of the sampling interval* is applied here, where the idea is to decrease its impact as the elapsed time increases.

The rest of the document is organized as follows. In section I, preliminaries on event-based control are introduced. The event-based LQR derived from [12] is recalled and extended to a version with dynamically varying (tunable) parameter. The integral action principle is also presented and the even-based LQR with integral action is finally detailed. Experimental results are then depicted in section II for the control of the angular positions of a gyroscope. They highlight the capabilities of the proposed approach and a significant reduction of the control updates. They also show that the actuator limits are guaranteed. Discussions finally conclude the paper.

I. EVENT-BASED LQR WITH INTEGRAL ACTION

A. Event-based (state-feedback) LQR

Let consider the linear time-invariant dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

$$\text{with} \quad x(0) := x_0$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^l$ are the state, (controlled) input and (measured) output vectors. System (1) is assumed to be stabilizable and x is measurable. Note that the dependence on t can be omitted in the sequel for the sake of simplicity.

Definition 1.1: By ***event-based state-feedback*** we mean a set of two functions:

i) an *event function* $\xi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, that indicates if one needs (when $\xi \leq 0$) or not (when $\xi > 0$) to recompute the control law,

ii) a *state-feedback function* $\mathbb{R}^n \to \mathbb{R}^m$, in the form

$$u(t) = -Kx(t) \tag{3}$$

where the state-feedback matrix K is calculated to make the closed-loop system stable.

The solution of system (1) with an event-based state-feedback starting in x_0 at $t = 0$ is then defined as the solution of the differential system

$$\dot{x}(t) = Ax(t) - BKx(t_i) \qquad \forall t \in [t_i, t_{i+1}[\tag{4}$$

where the time instants t_i, with $i \in \mathbb{N}$ (determined when the event function ξ vanishes) are considered as *events* and $x(t_i)$ is the memory of the state value at the last event.

In [12], it is proved that the linear system (1) can be asymptotically stabilized (as soon as (A, B) is a stabilizable pair) by means of a particular event-based state-feedback, defined by

$$u(t) = -Kx(t_i) \qquad \forall t \in [t_i, t_{i+1}[\tag{5}$$

$$\text{with} \quad K := 2\rho R^{-1} B^T P \tag{6}$$

$$\begin{aligned} \xi\big(x(t), x(t_i)\big) = (\sigma - 1)x(t)^T \big[PA + A^T P\big] x(t) \\ -4\rho x(t)^T PBR^{-1}B^T P\big[\sigma x(t) - x(t_i)\big] \end{aligned} \tag{7}$$

where $\rho > 0$ and $\sigma \in [0, 1[$ are tunable parameters, P, Q and R are positive definite matrices, with P solution of the algebraic Riccati equation (ARE) given by

$$PA + A^T P - 4\rho PBR^{-1}B^T P + Q = 0 \tag{8}$$

It is also proved in [12] that the feedback (5)-(7) is ***uniformly MSI*** (Minimal inter-Sampling Interval). That means it is a piecewise constant control with non zero sampling intervals, which is useful to avoid *Zeno* phenomena.

The event-based state-feedback (5)-(7) is afterwards called ***event-based LQR*** because it minimizes the value of a (infinite horizon) quadratic cost functional defined by

$$J = \int_0^\infty \left(x^T Q x + \rho u^T R u\right) dt \tag{9}$$

The first and second terms in (9) correspond to the energy of the controlled state (or output if $Q = C^T Q_y C$ using (2) for a given positive definite matrix Q_y) and input respectively, and the LQR strategy has to minimize both. However, decreasing the one requires the other is large, and inversely. The role of ρ consists in establishing a trade-off between these conflicting behaviors: the smaller ρ, larger is the control and smaller is the state. The role of Q (or Q_y) and R consists in weighting the different state (or output) and input variables. However, note that the feedback (5) is twice the classical optimal LQR feedback, as already noticed in [18]. For this reason, the particular choice $\rho = \frac{1}{2}$ will be applied hereafter for the same Q and R matrices in order to be able to then compare different closed-loop systems with equivalent control.

Let consider the function $V : \mathbb{R}^n \to \mathbb{R}$ defined by

$$V(x) := x^T P x \tag{10}$$

where the matrix P satisfies the ARE (8). This function is a control Lyapunov function (CLF) for system (1) since u in (5) renders $\dot{V}$ strictly negative for all $x \neq 0$. Actually, the idea behind the construction of the event-based feedback (5)-(7) is to compare the time derivative of the CLF V i) in the event-based case, that is when applying the piecewise constant state-feedback with $x(t_i)$ as in (5), and ii) in the classical case, that is applying the continuously varying state-feedback $x(t)$ instead of $x(t_i)$. The event function is the weighted difference between both, where σ is the weighted value. By construction, an event is enforced when the event function ξ vanishes to zero, that is hence when the stability of the event-based scheme does not behave as the one in the classical case. Also, the convergence will be faster with higher σ but with more frequent events in return. $\sigma = 0$ means updating the control when $\dot{V} = 0$.

Based on this behavior, σ is extended in the present paper to a ***dynamically varying parameter***. The idea is to have a faster convergence when the system trajectory is far from the desired setpoint (that is when the Lyapunov function V is high) and few events when it is almost stabilized. The proposed varying σ is defined by

$$\sigma\big(x(t), x(t_i)\big) := 1 - \delta \frac{V\big(x(t)\big)}{V\big(x(t_i)\big)} \tag{11}$$

where V is defined in (10) and $\delta \in]0, 1]$ is a new tunable parameter. Note that (11) guarantees the condition $\sigma \in [0, 1[$ since V is decreasing by construction of the event-based feedback (see [12] for the proof). As a consequence, the condition

$$V(x(t_i)) \geq \delta V(x(t)) > 0 \quad \forall x \neq 0,\, t \geq t_i \tag{12}$$

is satisfied if $\delta \in]0, 1]$. Moreover, a simple solution for δ (in order to not have anymore parameter to tune) is

$$\delta := \frac{V(x(t_i))}{V(x_0)} \tag{13}$$

where x_0 is the initial state value as defined in (1). The condition $\delta \in]0, 1]$ is also satisfied thanks to the decreasing of V. Note that the particular case $\delta = 0$ can occur in (13) for $x = 0$, that is when the system is stabilized, which is not in contradiction with the expected behavior (12). Nevertheless, the solution(13) will not be applied here.

B. Event-based output-feedback LQR

Whereas the full state information x is considered as measurable in a state-feedback approach (as in section I-A), in practice only a small number of values in the state vector (or linear combinations of the states) are really available in the output vector y. Therefore, the idea behind an output-feedback approach is to directly use the output in the control law, i.e. $u(t) = -\bar{K}y(t)$ where $\bar{K}$ is the output-feedback matrix for y as defined in (2), or to apply a state observer in order to have an estimation of the whole state information (this is possible as soon as (A, C) is an observable pair), and then build a state-feedback control law using the estimated state. This latter case is concerned here.

An extension of the event-based state-feedback (5)-(7) to an observer-based output-feedback version has been proposed in [7]. To summarize, assuming system (1)-(2) is observable and y is measurable, the typical *Luenberger* state observer for linear system is given by

$$\begin{aligned} &\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\big[y(t) - C\hat{x}(t)\big] \\ &\text{with} \quad \hat{x}(0) := \hat{x}_0 \end{aligned} \tag{14}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimated state vector. The matrix L is calculated to make stable the error of observation defined by

$$\tilde{x}(t) := x(t) - \hat{x}(t) \tag{15}$$

An optimal observer can be designed as the dual problem of the LQR state-feedback problem (also known as *Kalman filter*). The observer is optimal in estimating the state in the presence of zero-mean stochastic Gaussian processes corrupting the output measurements and the state, that is

$$\dot{x}(t) = Ax(t) + Bu(t) + w \tag{16}$$

$$y(t) = Cx(t) + v \tag{17}$$

where $v \in \mathbb{R}^l$ is the output noise and $w \in \mathbb{R}^n$ is the input perturbation. The cost to minimize can be expressed as

$$J = \int_0^\infty \left(e_w^T W e_w + \mu e_v^T V e_v\right) dt \tag{18}$$

where e_w and e_v are the errors of estimation in absence of noise ($v = 0$) and in absence of perturbation ($w = 0$) respectively. The role of $\mu > 0$, W and V positive definite matrices, then consists in establishing a trade-off between the quality of sensors (sensor noise, measurement bias) and quality of actuators (perturbations in the input, friction). By duality, the matrix L is finally obtained as

$$L := \mu U C^T V^{-1} \tag{19}$$

where U is positive definite solution of the ARE given by

$$AU + UA^T - 4\mu U C^T V^{-1} C U + W = 0 \tag{20}$$

Definition 1.2: By ***event-based output-feedback*** we mean a set of two functions:

i) an *event function* $\xi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (as defined above),
ii) an *output-feedback function* $\mathbb{R}^n \to \mathbb{R}^m$ in the form

$$u(t) = -K\hat{x}(t) \tag{21}$$

The solution of system (1)-(2) with an event-based output-feedback using the observer (14) and starting in x_0 at $t = 0$ is then defined as the solution of the differential system

$$\begin{aligned} &\dot{\hat{x}}(t) = A\hat{x}(t) - BK\hat{x}(t_i) + L\big[y(t) - C\hat{x}(t)\big] \\ &\dot{x}(t) = Ax(t) - BK\hat{x}(t_i) \qquad \forall t \in [t_i, t_{i+1}[\end{aligned} \tag{22}$$

Then, applying $\hat{x}$ instead of x in (5)-(7) makes asymptotically stable and *uniformly MSI* the closed-loop system for stable matrices K (defined as in (6)) and L (with L faster than K).

C. Trajectory tracking and integral action

Only null stabilization is considered in previous sections, which means all the state (and output) variables tend to reach zero. Nevertheless, it can be interesting the outputs track given setpoints and, accordingly, some of the state variables be nonzero in the steady state. The idea is hence to control the tracking errors (the errors between the states or outputs and their setpoints) instead of the states. Let define

$$x_e(t) := x(t) - x_r(t) \tag{23}$$

$$e(t) := r(t) - y(t) \tag{24}$$

where $x_r \in \mathbb{R}^n$ and $r \in \mathbb{R}^l$ are the state and output setpoint vectors, $x_e \in \mathbb{R}^n$ and $e \in \mathbb{R}^l$ are the state and output error vectors. Note that $r(t) := Cx_r(t)$ by construction. Rewriting system (1)-(2) with such a notation yields

$$\dot{x}_e(t) = Ax_e(t) + Bu(t) \tag{25}$$

$$\begin{aligned} &y(t) = Cx_e(t) \\ &\text{with} \quad x_e(0) := x_{e0} \end{aligned} \tag{26}$$

but this is known as not robust enough to disturbance.

A robust tracking can be achieved thanks to an external control loop and the so-called ***integral action***. Note that other methods can be applied, but an integral action also considers system disturbances because the error will converge even if the output responses are not as expected (because of model uncertainties or external perturbations for instance) [10]. The principle consists in building the extra states z defined as the integral of the error, that is

$$\begin{aligned} &\dot{z}(t) = e(t) \\ &\text{with} \quad z(0) := 0 \end{aligned} \tag{27}$$

and constructing the whole system when taking into account these added state variables. The augmented system, with new state vector

$$\bar{x}(t) := \begin{bmatrix} x_e(t)^T & z(t)^T \end{bmatrix}^T$$

where $\bar{x} \in \mathbb{R}^{n+l}$, becomes

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) + B_r r(t) \tag{28}$$

$$\begin{aligned} &y(t) = \bar{C}\bar{x}(t) \\ &\text{with} \quad \bar{x}(0) := \begin{bmatrix} x_{e0}^T & 0 \end{bmatrix}^T \end{aligned} \tag{29}$$

$$\bar{A} := \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \ \bar{B} := \begin{bmatrix} B \\ 0 \end{bmatrix}, \ \bar{C} := \begin{bmatrix} C & 0 \end{bmatrix}, \ B_r := \begin{bmatrix} 0 \\ I_l \end{bmatrix}$$

where I_l is the identity matrix of dimension l. Then, controlling the augmented system (28)-(29) with a state-feedback in the form of (3) – or an output-feedback in the form of (21) respectively – will make converge the augmented state $\bar{x}$ (and z in particular). As a consequence, the system outputs track the desired setpoint without static error. Note that the augmented state $\bar{x}$ has to be taken into account in the control law (5) but also in the event function (7), leading to augmented matrices $\bar{K}$, $\bar{P}$ and $\bar{Q}$ (R can remain the same since the number of inputs does not change).

In practice, a discrete-time (time-triggered) version of (27) (applying the backward difference approximation) is

$$z(t_k) = z(t_{k-1}) + \hbar e(t_k) \tag{30}$$

where $\hbar$ is the (constant) sampling period, t_k and t_{k-1} are two successive sampling instants, with $k \in \mathbb{N}$. However, in the event-based approach $\hbar$ becomes a varying interval, afterwards denoted $h_i := t_i - t_{i-1}$ where t_i are event instants with $i \in \mathbb{N}$. As a consequence, an integral action can induce overshoot issues when the sampling interval h_i becomes large since this variable is no more limited. This was notably demonstrated in [5] in the case of an event-based PI controller. Solutions to avoid such problems consist in bounding the integral gain (i.e. the product between h_i and $e(t_i)$) or adding an ***exponential forgetting factor of the sampling interval*** for instance. This latter method is dedicated here. The approach is somehow similar to the anti-windup mechanism used in control theory, where the error induced by the saturation has to be compensated. The integral action (30) becomes

$$\begin{aligned} &z(t_i) = z(t_{i-1}) + \lambda(h_i)e(t_i) \\ &\text{with} \quad \lambda(h_i) = h_i e^{\alpha(\hbar - h_i)} \end{aligned} \tag{31}$$

where α is a degree of freedom to increase/decrease the exponential sampling interval of the integral action. One can refer to [5] for further details. Therefore, the expression (31) will be applied instead of (30) in the *event-based LQR with integral action*. Note that other approaches proposed in [5] can also be implemented.

II. Experimental results: application to the gyroscope

Gyroscopes are widely used as actuators to control spacecrafts attitude for example. The physical principle consists in varying the rotational speed of a flying wheel (motorized gimbal) in order to apply a moment of controlled amplitude (variable-speed single-gimbal gyroscope) or to orientate the axis of the wheel (double-gimbal) to rotate the spacecraft. These devices are generally called control momentum gyroscopes (CMG) and have been a topic of prime interest in control theory.

A. Experimental platform

The experimental platform, depicted in Fig. 1, is a gyroscope M750p from ECP systems [1], where a (classical) LQR control has been previously investigated in [13].

1) Electromechanical plant: The gyroscope consists of 4 (rigid) rotating masses. The 4 rigid bodies each have as angular position θ_i relative to their rotating gimbal axis i, with $i = 1, 2, 3, 4$. More precisely, a high inertia brass rotor (body D) is suspended in an assembly with four angular degrees of freedom. The rotor spin torque is provided by a rare earth magnet type DC motor (motor 1) whose angular position (θ_1) is measured by an optical encoder (encoder 1). The first transverse gimbal assembly (body C) is driven by another rare earth motor (motor 2) to effect motion about axis 2. The relative position between bodies C and B (θ_2) is provided by another encoder (encoder 2). The subsequent gimbal assembly, body B, rotates with respect to body A about axis 3. There is no active torque applied about this axis and the relative angle (θ_3) is measured by encoder 3. Similarly, body A rotates without actively applied torque relative to the base frame (inertial ground) along axis 4. The relative angle (θ_4) is measured by encoder 4. Two manual brakes may be used to lock the relative position between either bodies A and B or body A and the base frame, in order to reduce the system degrees of freedom.

Fig. 1. ECP's gyroscope (model M750p).

2) Angular positions and torques: The gyroscope is assumed to be symmetric and the center of all rigid bodies (A, B, C et D) lie at the center of body D (the rotor). As a result, only the rotational dynamics needs to be taken into account. The following convention is adopted hereafter:

- The angular position θ_1 of the rotor (body D) is not of importance: only the angular velocity $\omega_1 = \dot{\theta}_1$ is considered.
- The angular position θ_2 of the rotor drum (body C) is set to 0 if the rotor drum (body C) is perpendicular to the inner gimbal (body B).
- The angular position θ_3 of the inner gimbal (body B) is 0 if the inner gimbal (body B) is perpendicular to the outer gimbal (body A).
- Since the outer gimbal (body A) is able to rotate freely and the gyroscope is assumed to be symmetric, θ_4 can be reset to $\theta_4 = 0$ at any angular position of the outer gimbal (body A).

The angular position of the 4 rigid bodies in the gyroscope can be controlled with the 2 internal torques T_1 and T_2. These torques are provided by DC motors: T_1 rotates D around its axis (flying wheel driver) while T_2 rotates the C body around the second axis (longitudinal).

3) Dynamic model: The gyroscope is a complex nonlinear system. However, for a constant angular velocity ω_1, it can be modelized as a multivariable linear system. Thus, considering small variations around the operating point defined by the angular speed $\omega_1 = \Omega$ and the angles $\theta_i = 0$ for $i = 2$ to 4, gives

$$\begin{aligned} \dot{\omega}_2 &= \frac{J_D\Omega}{I_C + I_D}\omega_4 + \frac{1}{I_C + I_D}T_2 \\ \dot{\omega}_3 &= -\frac{1}{J_B + J_C}T_1 \\ \dot{\omega}_4 &= -\frac{J_D\Omega}{I_D + K_A + K_B + K_C}\omega_2 \end{aligned} \tag{32}$$

The numerical values of the inertia of the four bodies are $K_A = 0.067kg.m^2$, $I_B = 0.012kg.m^2$, $J_B = 0.018kg.m^2$, $K_B = 0.030kg.m^2$, $I_C = 0.0092kg.m^2$, $J_C = 0.023kg.m^2$, $K_C = 0.022kg.m^2$, $I_D = 0.015kg.m^2$, $J_D = 0.027kg.m^2$. The (fixed) angular velocity for ω_1 is $\Omega = 42\,rad/s$. The actuators are limited to

$$\begin{aligned} |T1| &\leq 0.2\,Nm \\ |T2| &\leq 3.0\,Nm \end{aligned} \tag{33}$$

and the angles are limited to

$$|\theta_i| \leq 20^\circ \quad \text{for } i = 2 \text{ to } 4 \tag{34}$$

A state-space representation as given in (1)-(2) of (32) with state and input vectors respectively defined by

$$\begin{aligned} x &= \begin{pmatrix}\theta_3 & \theta_4 & \omega_2 & \omega_3 & \omega_4\end{pmatrix}^T \\ u &= \begin{pmatrix}T_1 & T_2\end{pmatrix}^T \\ y &= \begin{pmatrix}\theta_3 & \theta_4\end{pmatrix}^T \end{aligned} \tag{35}$$

then follows, with

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{J_D\Omega}{I_C+I_D} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{J_D\Omega}{I_D+K_A+K_B+K_C} & 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{I_C+I_D} \\ -\frac{1}{J_B+J_C} & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{36}$$

This system is in limit of stability, it is controllable and observable. An observer as defined in (14) is used in order to estimate the whole state vector.

Note that the objective of the given configuration (35) is not to control θ_2 which is stabilizable, but only θ_3 and θ_4.

B. Control parameters

In order to compare the *event-based LQR* with a (classical) time-triggered approach, the control parameters are tuned as in [13], that gives

$$\begin{aligned} &Q = C^TC, \quad R = \begin{bmatrix} 3 & 0 \\ 0 & 0.02 \end{bmatrix}, \quad \rho = \{1, \tfrac{1}{2}\} \\ &W = BB^T, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mu = 0.01 \end{aligned} \tag{37}$$

where W, V and μ are used for the observer's design as the dual parameters of Q, R and ρ respectively. The choice for ρ is either 1 for the time-triggered approach or $\frac{1}{2}$ for the event-based one (because the control is twice in this latter case, as already explained above). Note also that the particular choice for R takes into account the actuator limits (33), which should therefore be respected in practice. One can refer to [13] for further details.

For the *event-based LQR with integral action* strategy, augmented parameters are required for the control's design in order to fit with the augmented system (28)-(29). They are also as in [13], that yields

$$\bar{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{38}$$

in order to give priority on the stabilization of the extra states. Other parameters are unchanged.

For the event-based setups, different values for the tuning parameter σ will be applied, as well as the dynamically varying version proposed in (11)-(13) with $\delta = 0.2$ in this latter case. Different values for the parameter α will also be used in the exponential forgetting factor and $\hbar = 20\,ms$.

C. Performance indexes

Performance indexes, introduced in [15], are recalled here. They allow to compare the different event-based proposals with respect to classical approaches:

- The number (Nb) of samples required to perform the test bench.
- The IAE index, which gives information on the setpoint tracking
$$IAE = \int_0^\infty \big|e(t)\big|dt$$
- By analogy, the IAU index gives information on the control effort
$$IAU = \int_0^\infty \big|u(t)\big|dt$$

The performance indexes obtained for different experimental results are summarized in Table I. Results are discussed in the next section.

D. Experimental results

The gyroscope angular position θ_3 and θ_4 has to track a given (sinus signal) setpoint. Note that a first initializing part consists in rising the angular velocity ω_1 to the constant value Ω (thanks to a simple PI control on T_1) in order to satisfy the assumption required for the linear form (32). Only updates after this initialization are considered, that is after $t = 15\,s$. Different strategies are compared in the sequel:

i) Classical time-triggered (observer-based output-feedback) LQR:
   - with and without integral action.

TABLE I

PERFORMANCE INDEXES OBTAINED FOR THE DIFFERENT EXPERIMENTS WITH SEVERAL CONTROL STRATEGIES.

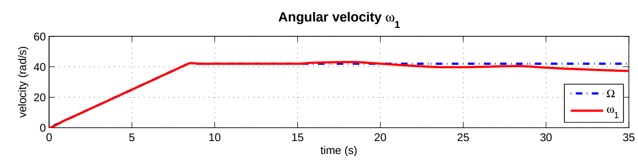
		Nb	IAE	IAU
Classical LQR strategy		1000	2.036	3.534
Event-based LQR strategy	$\sigma = 0.8$	84	1.812	3.723
	$\sigma = 0.5$	81	1.816	4.065
	$\sigma = 0.2$	69	1.806	4.046
	varying σ	96	1.876	3.750
Classical LQR with integral action		1000	1.189	3.940
Event-based LRQ with integral action with time-triggered integral action	$\sigma = 0.8$	99	2.318	3.550
	$\sigma = 0.5$	64	2.245	3.762
	$\sigma = 0.2$	55	2.303	3.343
	varying σ	92	2.231	3.653
Event-based LRQ with integral action without exponential forgetting factor	$\sigma = 0.8$	100	0.811	3.794
	$\sigma = 0.5$	86	1.023	4.461
	$\sigma = 0.2$	68	1.796	5.208
	varying σ	84	0.816	3.846
Event-based LRQ with integral action with exponential forgetting factor ($\alpha = 10$)	$\sigma = 0.8$	91	2.384	3.603
	$\sigma = 0.5$	61	2.362	3.479
	$\sigma = 0.2$	53	2.610	3.602
	varying σ	89	2.327	3.606
Event-based LRQ with integral action with exponential forgetting factor ($\alpha = 1$)	$\sigma = 0.8$	73	1.126	3.775
	$\sigma = 0.5$	50	1.284	4.017
	$\sigma = 0.2$	49	1.253	3.960
	varying σ	68	1.156	3.943

ii) Event-based (output-feedback) LQR:

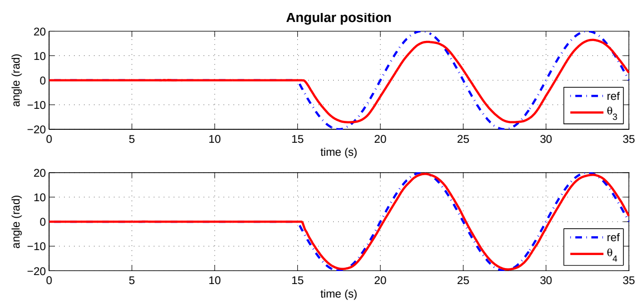
- with and without integral action;
- with and without exponential forgetting factor of the sampling interval (for the integral computing);
- with and without dynamically varying parameter σ.

Experimental results for the time-triggered LQR strategy without integral action are represented in Fig. 2. The top plot shows the variation of the angular velocity ω_1 which is considered as constant in the theoretical part. However, one can see that it is not. The initializing (ramp) part can be observed during the first $10\,s$, where the manual brakes are locked on axis 3 and 4. At $20\,s$, the brakes are unactivated and the LQR strategy is enabled. ω_1 hence varies whereas θ_3 and θ_4 are controlled. This plot is not repeated in the sequel. The bottom plots show the setpoints and the measured angles. Experimental results when adding the integral action are then represented in Fig. 3. The integral action ensures a better tracking of the angular positions with lower error (as one can see comparing the IAE index in Table I).

Several experimental results for the different event-based LQR strategies were realized, which performance indexes are all summarized in Table I. Performance are basically quite similar but with a strong decrease of control updates (more than $90\,\%$ in all the experiments). As before, the integral action allows to reduce the IAE index. In the event-based framework, this reduction seems related to the exponential forgetting factor (and its tuning parameter α). On the other hand, the frequency of control updates decreases with σ (as expected by construction of the event function) but with deteriorated IAE and IAU performance indexes in return. The dynamically varying version leads to a certain tradeoff between the number of updates and the performance. This is because the value of σ can increase in case of fast required samples and decrease when the system is naturally stable.

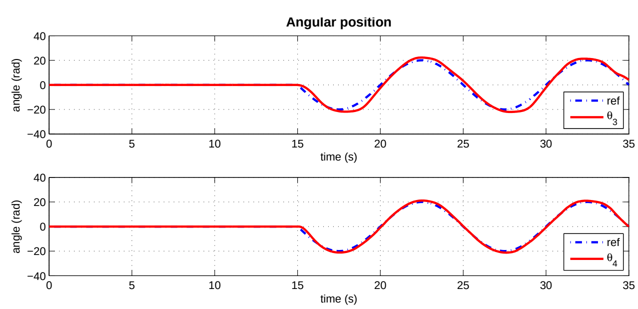


(a) Angular velocity ω_1.



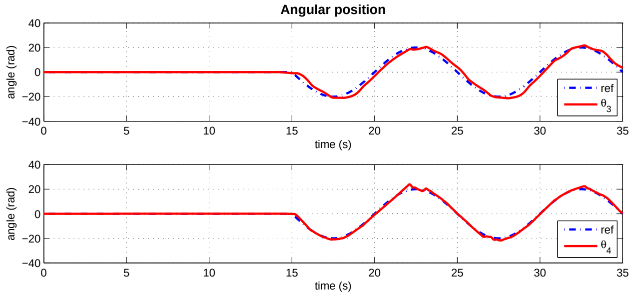
(b) Angles.

Fig. 2. Experimental results: conventional time-triggered LQR strategy without integral action.

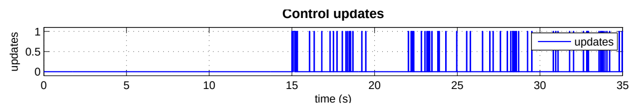


(a) Angles.

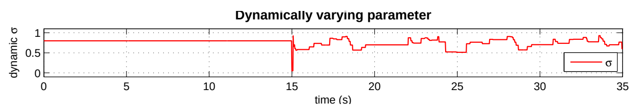
Fig. 3. Experimental results: conventional time-triggered LQR strategy with integral action.



(a) Angles.



(b) Updates.



(c) Dynamics of the parameter σ.

Fig. 4. Experimental results: event-based LQR strategy with integral action, exponential forgetting factor ($\alpha = 1$) and varying parameter σ.

Finally, experimental results of the event-based LQR with integral action, exponential forgetting factor ($\alpha = 1$) and varying parameter σ are represented in Fig. 4. Extra (bottom) plots are added to show the (no more constant) sampling instants (where '1' means the control law is calculated and updated during the sampling period $\bar{h}$, '0' means the control is kept constant) and the dynamics of parameter σ. This proposal gives even better results, with smaller IAE and quite close IAU indexes, but with about 93 % of samples less than the classical approach. Furthermore, the control signal satisfies the actuator limits of the system (not represented in the different plots) thanks to simple choice of the Q and R matrices, the same as in the classical LQR synthesis.

Conclusions and Future Works

This paper first recalled the *event-based LQR* scheme derived from the seminal work in [12]. Both state- and output-feedback cases were treated. A dynamically varying tunable parameter version was also suggested. Then, an *event-based LQR with integral action* strategy was developed and an *exponential forgetting factor of the sampling interval* added, based on previous work focusing on the integral gain in PI controllers [5]. The whole approach was finally tested on a real-time system, for the stabilization of the angular position of a gyroscope to a given setpoint. Experimental results showed the effectiveness of the proposal with a high reduction of the frequency updates for quite similar (or even better) performance. The actuator limits are also respected thanks to parameters tuning as simple as in a classical LQR strategy. The advantage of an event-driven scheme was hence highlighted and the encouraging results strongly motivate to continue developing event-based control strategies.

Next step is to work on dynamic event function. Another way of investigation could be to extend the proposal to nonlinear systems and consider delays, in the spirit of [12] and [6].

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